Solution: Consider the roots of the polynomial

$x^3 = 1$ i.e. $x^3 - 1 = 0$.

Its roots are $1, \frac{-1+\sqrt{3}i}{2}$.

let $\frac{-1+\sqrt{3}i}{2} = \omega$ and $\frac{-1-\sqrt{3}i}{2} = \omega^2$.

$\therefore$ $G = \{1, \omega, \omega^2\}$ under the binary operation '$\cdot$', where ω satisfies the ~~prop~~ following properties:

$$1 + \omega + \omega^2 = 0$$
$$\omega^3 = 1$$

$\cdot$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

Table 1.1

(i) Closure law: Since all the elements in Table 1.1 are elements of G, so Closure law holds in G.

(ii) Associative law: It can be easily checked that associative law holds in G, since these are elements in $\mathbb{C}$ = set of Complex numbers and associative law holds in $\mathbb{C}$ implies associative law holds in G.

(iii) Identity element: From table 1.1, it can be seen that '1' is two-sided identity as $1 \cdot a = a \ \forall a \in G$, by ~~first~~ second row of table and $a \cdot 1 = a \ \forall a \in G$, by ~~first~~ second column of table.

(iv) Existence of Inverse: Here identity element '1' occurs in each row. (in first column) If we see an element left to 1, and an element above 1 (in first row), then these are inverses of each other. i.e. element left to 1 is the left inverse of the element above 1. and an element above 1 is the right inverse of the element left to '1'.

i.e. $1 \cdot 1 = 1 = 1 \cdot 1$

$\omega \cdot \omega^2 = 1 = \omega^2 \cdot \omega$, ω

(V) Abelian: Since the entries in the composition table are symmetrical about the principal diagonal. Hence G is an abelian group under multiplication.

7.) (General linear Group of degree n).
Show that the set of all $n \times n$ matrices having non-zero determinant over the set of set of real numbers under the operation of matrix multiplication is a non-abelian group.

Solution: Let G be the set of all $n \times n$ non-singular matrices over the reals.

(i) Closure law: let $A, B \in G$

then AB is also a matrix of order $n \times n$.

and $\det(AB) = (\det A)(\det(B)) \neq 0$

$\therefore AB \in G$.

(ii) Associative law: Since matrix multiplication is associative, so associative law holds in G.

(iii) Existence of Identity: Let $I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$ be

the $n \times n$ identity matrix then $\det(I_n) = 1 \Rightarrow I_n \in G$.

Also $A \cdot I_n = A = I_n \cdot A$

$\therefore I_n$ is the identity element of G.

(iv) Existence of Inverse: Let $A \in G$ then A is non-singular matrix. Therefore, inverse of A exists.

Also A^{-1} is a $n \times n$ matrix and $\det A^{-1} = \frac{1}{\det A}$.

So $\det A^{-1} \neq 0$. $\therefore A^{-1} \in G$.

Such that $AA' = I_n = A'A$

$\therefore$ inverse of every element $A \in G$ exists

Hence all the properties of a group satisfied by G and so G is a group.

Note: Since, we know that matrix multiplication is not commutative, i.e. $AB \neq BA$.

$\therefore$ G is not a abelian group.

This group is known as General linear group of degree n and is denoted by $GL(n, R)$.

8) (Special linear group of degree n.)

Let G be the set of all $n \times n$ matrices over reals having determinant 1. Then G is a group under matrix multiplication.

Solution: (i) Closure law:

Let $A, B \in G$ then $\det A = 1$, $\det B = 1$.

$$\therefore \det AB = \det A \cdot \det B = 1 \cdot 1$$
$$= 1.$$

$\therefore$ $AB \in G$.

$\Rightarrow$ closure law holds in G.

(ii) Associative law: holds by matrix multiplication associative law.

(iii) Existence of Identity: Consider the identity matrix

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

then $\det(I_n) = 1$

$\Rightarrow$ $I_n \in G$.

Also $A \cdot I_n = A = I_n \cdot A$